Lecture 26

Change of variables.

Let $\Omega \subseteq \mathbb{R}^n$ open, $F: \Omega \to \mathbb{R}^n$ a $\mathcal{C}^1$-map ($\frac{\partial F}{\partial x_j} = F_{x_j}$ exist + cont.). Let DF_x denote linear map $\mathbb{R}^n \to \mathbb{R}^n$ given by matrix of partial derivatives at x. If $\det DF_x \neq 0$, $\forall x \in \Omega$, then by Inverse Function Thm, $F(\Omega) \subseteq \mathbb{R}^n$ is open.

<u>Def.</u> F is a <u>diffeomorphism</u> $\Omega \to \Omega'$ if $\Omega' = F(\Omega)$ and F is invertible with F^{-1} being $\mathcal{C}^1$.

<u>Rem.</u> By IFT, suffices that $\det DF_x \neq 0$ and F injective.

Thm 1. Let $\Omega \subseteq \mathbb{R}^n$ be open, $F: \Omega \to \Omega' = F(\Omega)$ a diffeomorphism.

(i) $E \in \mathcal{L}^n \Rightarrow F(E) \in \mathcal{L}^n$ and

$$m(F(E)) = \int_E |\det DF| \, dm$$

(ii) f $\mathcal{L}^n$-meas. on $\Omega' \Rightarrow f \circ F$ is $\mathcal{L}^n$-meas. on Ω. If $f \in L^+$ or L^1

$$\int_{\Omega'} f \, dm = \int_{\Omega} (f \circ F) \, |\det DF| \, dm$$

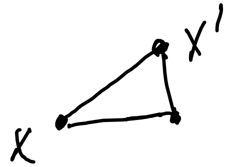
Pf. By similar arguments to the pf of Thm 1, suffices to prove the result for $\mathcal{B}_{\mathbb{R}^n}$ and $\mathcal{B}_{\mathbb{R}^n}$-meas. f.

We shall need the following:

Lemma 1. Let $U \subseteq \mathbb{R}^n$ open. Then $\exists$ $\{R_k\}_{k=1}^{\infty}$ closed equilateral cubes R_k w/ disjoint interiors, s.t. $U = \bigcup_{k=1}^{\infty} R_k$.

Pf. Construction using $A_i(U)$ from Lecture 23 on Jordan content. See Lemma 2.43 for details. ∎

So, what happens to cubes R? If we write $F=(F_1,\dots,F_n)$, then by Mean Value Thm; $x,x' \in R \Rightarrow$



$$|F_j(x)-F_j(x')| \le \sum_{k=1}^{n} \sup_R |F_{j,x_k}|\,|x_k-x'_k|$$

$$\le \sup_R \|DF\| \cdot \|x-x'\|_\infty$$

if $\|x-x'\|_\infty = \max_{1\le k\le n} |x_k-x'_k|$, $\|\underset{n\times n}{T}\| = \max_{1\le i\le n} \sum_{j=1}^{n} |T_{ij}|$

$$\Rightarrow \|F(x)-F(y)\|_\infty \le \sup_R \|DF\| \cdot \|x-y\|_\infty .$$

Thus, if $R = \underbrace{\{\|x-x_0\|\le r\}}_{\text{cube}}$, then

$$F(R) \subseteq \tilde{R} = \{\|y-F(x_0)\| < \sup_R \|DF\| \cdot r\}$$

$$\Rightarrow \quad m(F(R)) \le \left(\sup_R \|DF\|\right)^n \cdot m(R). \quad (*)$$

Since we can exhaust Ω by Ω_j open s.t. $\overline{\Omega}_j \subset\subset \Omega_{j+1}$ and $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$, it suffices

compactly contained

to assume that $\overline{\Omega} \subset\subset \mathbb{R}^n$ and $F \in C^1(\overline{\Omega})$.

(General case follows from MCT.)

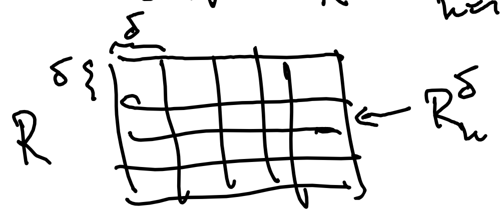
By uniform cont. of DF, $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. if $R = \{\|x - x_0\| \leq \delta\} \subseteq \Omega$ then

$$\sup_R \|\underbrace{(DF_{x_0})^{-1} \cdot (DF_x)}_{= I \text{ for } x = x_0}\|^n < (1+\varepsilon)$$

$\Rightarrow$ By Thm 1 w/ $T = DF_{x_0}$ and $(*)$

$$m(F(R)) = |\det DF_{x_0}|\, m((DF_{x_0})^{-1}(F(R)))$$
$$\leq |\det DF_{x_0}|\,(1+\varepsilon)\, m(R).$$

Now, if R is any cube, we can subdivide it $R = \bigcup_{n=1}^{m_\delta} R_n^\delta$



Since the sides of a cube have $m^n = 0$, we get

$$m(F(R)) \le (1+\varepsilon) \sum_{k=1}^{m_\delta} |\det DF_{x_k}|\, m(R_k^\delta)$$

$$= (1+\varepsilon) \int_R \varphi_\delta \, dm,$$

where $\varphi_\delta = \sum_{k=1}^{m_\delta} |\det DF_{x_k}| \chi_{R_k^\delta}$.

Pick a sequence $\varepsilon_k = \frac{1}{k} \searrow 0$, $\delta_k = \delta(\varepsilon_k) \to 0$.
Then $\varphi_k = \varphi_{\delta_k} \to |\det DF_x|$ pointwise (unif. actually by cont. of $\int$) and $|\varphi_k| \le \sup_R |\det DF_x| \in L^1(R, m)$. DCT $\Rightarrow$

$$m(F(R)) \le \int_R |\det DF_x| \, dm \quad (**)$$

If $U \subseteq \Omega$ is any open set, then $U = \bigcup_{k=1}^{\infty} R_k$, where R_k are cubes w/ disjoint interiors.

By (**), (again using $m^n=0$ for sides)

$$m(F(U)) \le \int_U |\det DF_x|\, dm .$$

Now, if $E \subseteq \Omega$ is any Borel set, then $\exists$ G_δ-set $G = \bigcap_{j=1}^{\infty} U_j$ s.t. $E \subseteq G$ and $m(E) = m(G)$ $(m(G \setminus E) = 0)$. Since $m(\Omega) < \infty$ (by assumption above), cont. from above $\Rightarrow$

$$m(F(E)) \le m(F(G)) = \lim_{j\to\infty} \int_{U_j} |\det DF_x|\, dm$$

$$= \int_G |\det DF_x|\, dm = \int_E |\det DF_x|\, dm$$

$$\Rightarrow \int_{\Omega' = F(\Omega)} f\, dm \le \int_\Omega (f \circ F)\, |\det DF_x|\, dm \quad (\dagger)$$

for all $f = \chi_{F(E)}$ $\Rightarrow$ for all simple function in L^+ $\Rightarrow$ for all $f \in L^+$ "Standard arg."

But since F is diffeomorphism, we can apply this to the inverse F^{-1} and $(f \circ F)|\det DF_x|$ on Ω

$$\int_{\Omega} (f \circ F)|\det DF_x|\, dm \leq$$

$$\int_{F(\Omega)} f \underbrace{|\det DF_x \circ F^{-1}| \cdot |\det DF_x^{-1}|}_{\det D(\underbrace{F \circ F^{-1}}_{\text{Identity}})_x} dm$$

$$= \int_{F(\Omega)} f\, dm.$$

Thus, if $f \in L^+ \Rightarrow \int_{F(\Omega)} f\, dm = \int_{\Omega} (f \circ F)|\det DF_x|\, dm$

$\Rightarrow$ same for $f \in L^1$ $(f = f^+ - f^-)$.

$\Rightarrow$ (ii). Since (ii) $\Rightarrow$ (i) we are done (assuming f $\mathcal{B}_{\mathbb{R}^n}$-meas. + $m(\Omega) < \infty$. $\mathcal{L}^n$-meas. follows as before and general Ω by σ-finite. arg.). ☒